Tolerance Relation Based Granular Space*

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Abstract. Granular computing as an enabling technology and as such it cuts across a broad spectrum of disciplines and becomes important to many areas of applications. In this paper, the notions of tolerance relation based information granular space are introduced and formalized mathematically. It is a uniform model to study problems in model recognition and machine learning. The key strength of the model is the capability of granulating knowledge in both consecutive and discrete attribute space based on tolerance relation. Such capability is reestablished in granulation and an application in information classification is illustrated. Simulation results show the model is effective and efficient.

1 Introduction

Information granules, as the name itself stipulates, are collections of entities, usually originating at the numeric level, that are arranged together due to their similarity, functional adjacency, indistinguishability, coherency or alike [1]. The entities on data layer usually belong to two types: discrete or consecutive. Many models and methods of granular computing [2][3][4][5][11][12] have been proposed and studied, however, most of them discuss discrete and consecutive data respectively. In their theories, discretization features are represented by attributes, which is calculated by the methods such as feature extraction, feature reduction and classification or only discretization. That means the features of one type can be generated from the other. So, it is time to construct a uniform model to study some important problems in pattern recognition and machine learning, such as feature extraction, feature reduction, discretization and classification.

Nowadays, many researchers study the equivalence relation based granular computing theory, such as Zhang B. and Y.Y. Yao [3][6] indicate that granule is closely related to quotient space. In reality, tolerance relation is a more broad relation. So, this paper discuss mainly about the tolerance relation based granular computing theory. There are a lot of papers on tolerance based rough set approaches [13][14][15], but this approaches don't uses the multi-level framework for granular computing and discuss mainly discretization features.

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2 Model of Tolerance Relation Based Granular Space

In 1962, Zeeman proposed that cognitive activities can be viewed as some kind tolerance spaces in function spaces. The tolerance spaces, which are constructed by tolerance relations based on distance functions, is used for stability analysis of dynamic system by Zeeman. In this paper, a tolerance spaces based on distance functions are developed for the analysis of information granulation, which is defined as tolerance relation granulation in the following parts.

2.1 Tolerance Relation Based Granular Space

The aim of describing a problem at different granularities is to enable the computer to solve the same problem at different granule size hierarchically. Suppose the triplet ($\boldsymbol{OS}$, $\boldsymbol{TR}$, $\boldsymbol{NTC}$) describes a tolerance relation based granular space $\boldsymbol{TG}$, where

$\boldsymbol{OS}$ denotes an object set system, which is illustrated by definitions 2.1-2.2;
$\boldsymbol{TR}$ denotes a tolerance relation system, which is illustrated by definitions 2.3-2.7;
$\boldsymbol{NTC}$ denotes a nested tolerance covering system, which is illustrated by definitions 2.8-2.13.

2.2 Object Set System

Object set system is composed by the objects at difference levels. $\boldsymbol{OS}_k$ represents an object at level $\boldsymbol{k}$.

Definition 2.1. $\boldsymbol{OS}_0$, called an original object vector, is a vector of $\boldsymbol{R}^n$, where $\boldsymbol{R}$ is the real number set.

Definition 2.2. $\boldsymbol{OS}_1$, called a subset object of level 1, is a set of original object vectors. Generally speaking, $\boldsymbol{OS}_{k+1}$ is a set of level $\boldsymbol{k}$ subset objects, $\boldsymbol{OS}_k$.

For example, in image processing, $\boldsymbol{OS}_0$ can be viewed as a pixel of an image, $\boldsymbol{OS}_1$ can be viewed as an image and $\boldsymbol{OS}_2$ can be viewed as a set of frames in video stream.

2.3 Tolerance Relation System

Tolerance relation system is a (parameterized) relation structure, and it is composed by a set of tolerance relations.

Definition 2.3. A tolerance relation $\boldsymbol{sn}$, $\boldsymbol{sn}\subseteq\boldsymbol{X}\times\boldsymbol{X}$, is a reflexive and symmetrical binary relation, where $\boldsymbol{X}$ is the original space of object vector and $\boldsymbol{X}\subseteq\mathbf{R}_n$.

Suppose α and β are two $\boldsymbol{n}$ dimensional vectors of $\boldsymbol{X}$, and $\boldsymbol{dis}(\alpha,\beta|\omega)$ is a distance function, where the dimensional weight $\omega = (\omega_0, \omega_1, !`, \omega_{n-1})$ and $\omega_i \geq 0$.

Definition 2.4. $\boldsymbol{sp}(\alpha,\beta|\boldsymbol{dis},\boldsymbol{d})$, called a simple tolerance proposition, is defined as

$$\boldsymbol{sp}(\alpha,\beta|\boldsymbol{dis},\boldsymbol{d}) \Leftrightarrow \boldsymbol{dis}(\alpha,\beta|\omega) \leq \boldsymbol{d}, \quad (1)$$

where $\boldsymbol{d}\geq 0$ is a real number, called the radius of $\boldsymbol{sp}(\alpha,\beta|\boldsymbol{dis},\boldsymbol{d})$.

Definition 2.5. A compound tolerance proposition $\boldsymbol{P}(\alpha, \beta|\boldsymbol{D})$, where $\boldsymbol{D} = \{d_1, d_2, \cdots, d_k\}$ and d_i is the radius of $\boldsymbol{sp}_i(\alpha, \beta|\boldsymbol{dis}_i, d_i)$, is a Boolean function composed by a group of $\boldsymbol{sp}_i(\alpha, \beta|\boldsymbol{dis}_i, d_i)$ related with "$\wedge$" , "$\vee$" and "$\neg$" operators and $0 \leq \boldsymbol{i} \leq \boldsymbol{k}$. For simplicity, the dimensional weight ω_i in $\boldsymbol{sp}_i(\alpha, \beta|\boldsymbol{dis}_i, d_i)$ is same.

In the case that $\boldsymbol{P}(\alpha, \beta|\boldsymbol{D})$ contains the negative operator "$\neg$", $\boldsymbol{P}(\alpha, \beta|\boldsymbol{D})$ may not be reflexive, and for $\boldsymbol{sp}(\alpha, \beta|\boldsymbol{dis}, \boldsymbol{d}) \Leftrightarrow \boldsymbol{dis}(\alpha, \beta|\omega) \geq \boldsymbol{d}$ is not a tolerance relation, so it can't be used in compound tolerance proposition. In this case, $\boldsymbol{P}(\alpha, \beta|\boldsymbol{D})$ can be recomposed by extending it to $\boldsymbol{P}(\alpha, \beta|\boldsymbol{D}) \vee (\boldsymbol{dis}(\alpha, \beta|\omega) \leq \boldsymbol{d})$, where $\boldsymbol{d}_0 \leq \boldsymbol{d}$.

Definition 2.6. The tolerance relation $\boldsymbol{sn}(\boldsymbol{P}, \omega, \boldsymbol{DIS}, \boldsymbol{D})$ induced by $\boldsymbol{P}(\alpha, \beta|\boldsymbol{D})$ is defined as $(\alpha, \beta) \in \boldsymbol{sn}(\boldsymbol{P}, \omega, \boldsymbol{DIS}, \boldsymbol{D}) \Leftrightarrow \boldsymbol{P}(\alpha, \beta|\boldsymbol{D})$, where $\boldsymbol{DIS} = \{\mathbf{dis}_1, \boldsymbol{dis}_2, \cdots, \boldsymbol{dis}_k\}$.

Proposition $\boldsymbol{P}$, weight vector ω, distance function vector $\boldsymbol{DIS}$ and radius vector $\boldsymbol{D}$ are the four important elements in a tolerance relation. Tolerance relation system is composed by a set of tolerance relations and many space areas can be described by tolerance relation system.

2.4 The Nested Tolerance Covering System

The nested tolerance covering system is a (parameterized) granule structure, which denotes different levels granules and the granulation process based on above object system and tolerance relation system. The nested tolerance covering on $\boldsymbol{OS}_k$ can be constructed as follows.

The Nested Tolerance Covering on $\boldsymbol{OS}_1$

In this subpart, the definitions of granules, tolerance covering and nested tolerance covering are presented. Besides, with definition 2.8 and definition 2.9, the granulation process on $\boldsymbol{OS}_1$ is illustrated. Here, we focus on the extension of a granule, that is, how to use the objects to construct granules.

Definition 2.8. A small granule over $\boldsymbol{OS}_1$ is a set

$$\boldsymbol{G}_0(\boldsymbol{a}\,|\omega_0) = \{\boldsymbol{x}\,|(\boldsymbol{x}, \boldsymbol{a}) \in \boldsymbol{sn}(\boldsymbol{P}, \omega_0, \boldsymbol{DIS}, \boldsymbol{D}) \wedge \boldsymbol{x} \in \boldsymbol{OS}_1\}, \tag{2}$$

where $\boldsymbol{Grid} \subseteq \boldsymbol{OS}_1$ and $\boldsymbol{a}$=(a'), a'$\in \boldsymbol{Grid}$. a' can be viewed as the location of $\boldsymbol{G}_0(\boldsymbol{a}|\omega_0)$. $\boldsymbol{Grid}$ is the set of all possible locations and defined as grid point set. ω_0 is the coordinate.

Definition 2.9. A nested tolerance covering over $\boldsymbol{OS}_1$ is defined as follows:

(1) A level 0 granule $\boldsymbol{G}_0(\boldsymbol{a}|\omega_0)$ is a subset of $\boldsymbol{OS}_1$ under coordinate $\boldsymbol{L}_0 = \omega_0$ and $\boldsymbol{a}$ is the location of $\boldsymbol{G}_0(\boldsymbol{a}|\omega_0)$ in $\boldsymbol{OS}_1$. The set of all level 0 granules, $\{\boldsymbol{G}_0(\boldsymbol{a}|\omega_0)\}$, under L_0, a grid point set $\boldsymbol{Grid}_0$ and a tolerance relation set $\boldsymbol{sn}(\boldsymbol{L}_0, \boldsymbol{Grid}_0)$ is defined as $\boldsymbol{C}_1(0)$.

(2) Suppose $\boldsymbol{G}_k(\eta_k|\omega_k)$ is a level $\boldsymbol{k}$ granule and a level $\boldsymbol{k}$+1 granule

$$\boldsymbol{G}_{k+1}(\eta_{k+1}|\omega_{k+1}) = \{\boldsymbol{x}\,|((\boldsymbol{x}, \eta_{(k+1)(k+1)}) \in \boldsymbol{sn}(\boldsymbol{P}, \omega_{k+1}, \boldsymbol{DIS}, \boldsymbol{D})) \wedge \boldsymbol{x} \in G_k(\eta_k|\omega_k)\} \tag{3}$$

where η_i=(η_{i0} η_{i1}, $\cdots$, η_{ii}). η_i is the location set of all ancestor granules of $\boldsymbol{G}_i(\eta_i|\omega_i)$. η_{ip} is the location of $\boldsymbol{G}_i(\eta_i|\omega_i)$'s ancestor $\boldsymbol{G}_p(\eta_p|\omega_p)$ in its father granule $\boldsymbol{G}_{p-1}(\eta_{p-1}|\omega_{p-1})$. For $\boldsymbol{G}_0(\eta_0|\omega_0)$, $\eta_0 = \boldsymbol{a}$. $\eta_{(k+1)(k+1)}$ is the location of $\boldsymbol{G}_{k+1}(\eta_{k+1}|\omega_{k+1})$ in its father granule $\boldsymbol{G}_k(\eta_k|\omega_k)$. If $\boldsymbol{G}_{k+1}(\eta_{k+1}|\omega_{k+1})$=$\boldsymbol{G}_k(\eta_k|\omega_k)$, the set of all small level $\boldsymbol{k}$+1 granule $\boldsymbol{G}_{k+1}(\eta_{k+1}|\omega_{k+1})$ is defined as tolerance covering $\boldsymbol{GW}_{k+1}(\eta_{k+1}|\omega_{k+1})$ on $\boldsymbol{G}_k(\eta_k|\omega_k)$, which is based on the coordinate system $\boldsymbol{L}_{k+1} = (\omega_0, \cdots, \omega_{k+1})$, a grid point set $\boldsymbol{Grid}_{k+1} \subseteq \boldsymbol{G}_k(\eta_k|\omega_k)$, and a tolerance relation set $\boldsymbol{sn}(\boldsymbol{L}_{k+1}, \boldsymbol{Grid}_{k+1})$.

Based on above, suppose $\boldsymbol{C}_1(0)$=$\boldsymbol{GW}_0(\eta_0|\omega_0)$, $\boldsymbol{C}_1(\boldsymbol{k})$={$\boldsymbol{GW}_k(\eta_k|\omega_k)$}, and $\bigcup \boldsymbol{GW}_k(\eta_k|\omega_k)$= $\boldsymbol{OS}_1$, then

$$\bigcup_{\boldsymbol{k}=0,1,\cdots} \boldsymbol{TC}_1 = (\boldsymbol{C}_1(\boldsymbol{k})) \tag{4}$$

is the nested tolerance covering over $\boldsymbol{OS}_1$.

The Adjoint Nested Tolerance Covering System on Level *k* Granules
In this subpart, the definition of adjoint subset object are presented, which can be viewed as the intension of a granule. Two ways to generate the adjoint subset object are developed as follows.

Definition 2.10. An adjoint subset object at level $\boldsymbol{k}$ over a nested tolerance covering of $\boldsymbol{OS}_1$ is a new feature vector set. Each new feature vector belongs to a level k granule $\boldsymbol{G}_k(\eta_k|\omega_k)$, and can be generated from nested smaller granules by two ways:

(1) Computing a new vector directly over all original object vectors $\boldsymbol{OS}_0$ (Def.2.1) belong to $\boldsymbol{G}_k(\eta_k|\omega_k)$. For example, the vector can be the centroid vector of $\boldsymbol{G}_k(\eta_k|\omega_k)$, or a new long vector constructed by arranging all vectors of $\boldsymbol{G}_k(\eta_k|\omega_k)$ in a row, which is a prevalent method in image module matching algorithms.

(2) Computing a new vector through nested granules in $\boldsymbol{C}_1(\boldsymbol{k})$. According to above defined nested structure, a level $\boldsymbol{k}$ granule is larger than a level $\boldsymbol{k}$+1 granule, so new vectors at level $\boldsymbol{k}$ can be calculated from new vectors of level $\boldsymbol{k}$+1, each object vector $\boldsymbol{OS}_0$ in $\boldsymbol{OS}_1$ can be viewed as the highest level new vectors, and $\boldsymbol{OS}_1$ itself can be viewed as a level 0 granule and assigned a new feature vector. After assigning every level $\boldsymbol{k}$ granule of $\boldsymbol{OS}_1$ a new feature vector, all level $\boldsymbol{k}$ granules can be viewed as a new subset object.

The first kind is called as a usual adjoint subset object and the second is called as a nested subset object. Based on adjoint subset objects, adjoint nested tolerance covering system can be created as:

Definition 2.11. After assigning a new feature vector $\boldsymbol{V}$ to every $\boldsymbol{OS}_1$, a nested tolerance covering

$$\boldsymbol{TC}_2 = (\bigcup_{k=0,1,\cdots} \boldsymbol{C}_2(\boldsymbol{k})) \tag{5}$$

on $\boldsymbol{OS}_2$ can be constructed using the method constructing $\boldsymbol{TC}_1$ in $\boldsymbol{OS}_1$. A granule of $\boldsymbol{TC}_2$ can be viewed as a classification of $\boldsymbol{OS}_1$ and an integral class label $\boldsymbol{c}$ can be assigned to $\boldsymbol{OS}_1$. If we divide every dimension of feature vectors $\boldsymbol{V}$ into regions marked by $\boldsymbol{n}\in\{0,1,2,\cdots, \mathrm{N}\}$, then a new discretized feature vector $\boldsymbol{V}$' can be created by adding the classification label $\boldsymbol{c}$ to it. Such kind of new feature vectors are denoted as decision objects, ***Iobject***.

Similarly, we can define $\boldsymbol{TC}_k$ on $\boldsymbol{OS}_k$ and decision objects on $\boldsymbol{OS}_k$. The tolerance relation based granular space is so versatile that it includes all classification processes using distance functions and most of the multi-scale feature extraction processes in pattern recognition. For the sake of pages, we only focus on the knowledge discovery of lattice sub space of above granular space.

3 The Lattice Sub Space in Granular Space

Lattice is a simple but important sub space structure included by above granular space.

Definition 3.1. A level l granule $\boldsymbol{G}_l(\eta|\omega)$ on a subset object $\boldsymbol{OS_p}$ is a set of $\boldsymbol{OS}_{p-1}$, so in some cases, there are lattices $\boldsymbol{L} \subseteq \boldsymbol{TC}_p$, where $\boldsymbol{TC}_p = \bigcup(\boldsymbol{C}_p(\boldsymbol{k}))$ is a nested tolerance covering on $\boldsymbol{OS}_p$. These lattices are based on inclusion relation "$\subset$" and operators "$\cup$" and "$\cap$". In the lattice case, there must be some overlapped granules in $\boldsymbol{C}_p(\boldsymbol{l})$. If none of granules in $\boldsymbol{TC}_p$ are overlapped, granules in $\boldsymbol{C}_p(\boldsymbol{l})$ can be viewed as equivalent classes, so the granular theory based on equivalent relation is a special case.

Many problems can be described by the lattice space. In the following pages, we illustrate an example of lattice space, which is based on decision objects. Our object is to classify new decision objects according to the knowledge extracted from old classified objects. The information lattice space, based on a nested tolerance relation based granular space, can be used to model problems and describe problem solving algorithms. We also take a lot of experiments to test our theory's validity and ability to this problem.

3.1 The Construction of Tolerance Relation Based Granular Space

First, we define the object set system. Here $\boldsymbol{OS}_0 \in \boldsymbol{OS}_1$ can be viewed as a decision object $\boldsymbol{Iobject} = (\boldsymbol{v}_0, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_{n-1}, \boldsymbol{v}_n)$, where $\boldsymbol{v}_i$ is a discretized feature and $\boldsymbol{v}_n = \boldsymbol{c}$ is the class label of this decision object. $\boldsymbol{OS}_1$ can also be viewed as a decision table (**Def.3.7**) composed by decision objects.

Second, we define the tolerance relation system. The distance function $\boldsymbol{dis}(\alpha, \beta| \omega_i)$ and the tolerance proposition $\boldsymbol{P}(\alpha, \beta|\boldsymbol{D})$ can be defined by the following definition.

Definition 3.2. $\boldsymbol{P}(\alpha, \beta|\boldsymbol{D})$=$\boldsymbol{dis}(\alpha, \beta|\omega) \leq 0$, where

$$\boldsymbol{dis}(\alpha, \beta|\omega_i) = \sum_{\boldsymbol{j}=0}^{\boldsymbol{n}} \omega_{ij}(\alpha_j \oplus \beta_j), \tag{6}$$

$\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n), \beta = (\beta_0, \beta_1, \cdots, \beta_n)$, and

$$\alpha_j \oplus \beta_j = \begin{cases} 0 & \text{if } \alpha_j = \beta_j \\ 1 & \text{else} \end{cases} \tag{7}$$

According to above definition, the tolerance relation is generated as $(\alpha, \beta) \in \boldsymbol{sn}(\boldsymbol{P}, \omega, \boldsymbol{DIS}, \boldsymbol{D}) \Leftrightarrow \boldsymbol{P}(\alpha, \beta|\boldsymbol{D})$. ω_i=(ω_{i0}, ω_{i1},$\cdots$,ω_{in}) is a coordinate and ω_{ij}=1 or 0.

Third, we define the nested tolerance covering system. The following definition presents the method generating a new granule and its adjoint vector. With different forms of coordinate ω, different new granules are generated. The last element of ω is always "1" because the granules without the same decision cannot be combined.

Definition 3.3. A nested tolerance covering $\boldsymbol{TC}_1 = (\bigcup_{k=0,1,\cdots} \boldsymbol{C}_1(\boldsymbol{k}))$ on $\boldsymbol{OS}_1$ can be constructed using the method defined in section 2.4.1, where

$$\boldsymbol{C}_1(\boldsymbol{i}) = \{\boldsymbol{G}_i(\boldsymbol{Ob}_j|\omega_i)|\boldsymbol{G}_i(\boldsymbol{Ob}_j|\omega_i) = \{\boldsymbol{Ob}_l|\boldsymbol{dis}(\boldsymbol{Ob}_l, \boldsymbol{Ob}_j|\omega_i) \le 0,\ \boldsymbol{Ob}_j \in \boldsymbol{OS}_1\}\}, \tag{8}$$

where $\boldsymbol{Ob}_j = (v_{j0}, v_{j1}, \cdots, v_{jn})$. $\boldsymbol{G}_i(\boldsymbol{Ob}_j|\omega_i)$ is a level $\boldsymbol{i}$ granule and $\boldsymbol{i}$=$\Sigma_{j=0,\cdots,n-1}$ ω_{ij}. A new vector $\boldsymbol{VG}_{ij}$=($\boldsymbol{VG}_{ij0}$, $\boldsymbol{VG}_{ij1}$, $\cdots$, $\boldsymbol{VG}_{ij(n-1)}$, $\boldsymbol{VG}_{ijn}$) , denoted as decision rule (Def.3.9), is assigned to it, where if ω_p is equal to 1 then vg_{ijp}= v_{jp}, else vg_{ijp}="*". If $\boldsymbol{dis}(\boldsymbol{Ob}_l, \boldsymbol{Ob}_j|\omega_i) \le 0$, then $\boldsymbol{G}_i(\boldsymbol{Ob}_l|\omega_i) = \boldsymbol{G}_i(\boldsymbol{Ob}_j|\omega_i)$.

For example, if $\boldsymbol{Ob}_1 = (\boldsymbol{f}_1, \boldsymbol{r}_1, \boldsymbol{e}_1, \boldsymbol{s}_1)$, $\boldsymbol{Ob}_2 = (\boldsymbol{f}_1, \boldsymbol{r}_1, \boldsymbol{e}_2, \boldsymbol{s}_1)$, $\boldsymbol{r} = 3$, and ω=(1, 1, 0, 1), then $\boldsymbol{G}_2(\boldsymbol{Ob}_2) = (\boldsymbol{f}_1, \boldsymbol{r}_1, *, \boldsymbol{s}_1)$, and the level of the granule is $\boldsymbol{i}$=2.

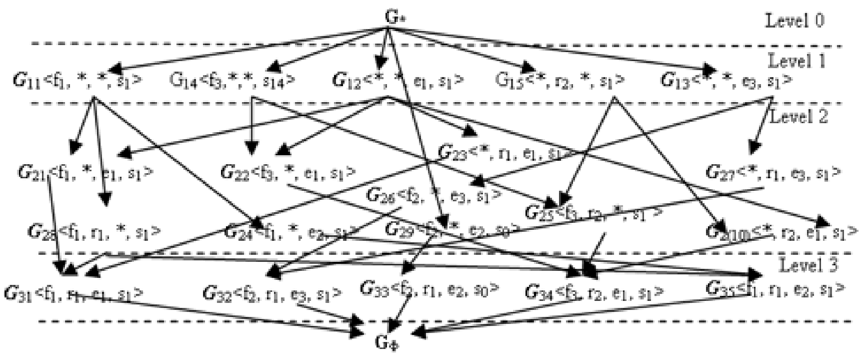

Fig. 1. Decision Granular Lattice of Table 1

In our algorithm, $\boldsymbol{VG}_{ij}$ is computed from $\boldsymbol{VG}_{(i+1)l}$ with method 2 in definition 2.10 and the following definition is more detailed one. **Definition 3.4.** Suppose $\boldsymbol{C}_1(\boldsymbol{i}) = \{\boldsymbol{G}_{i1}, \boldsymbol{G}_{i2}, \cdots, \boldsymbol{G}_{iq}\}$ is a level $\boldsymbol{i}$ tolerance covering over $\boldsymbol{G}_{(i-1)p} \subseteq \boldsymbol{OS}_1$. By Proposition 3.2, $\boldsymbol{q}$=2 and $\boldsymbol{dis}(\boldsymbol{VG}_{i1}, \boldsymbol{VG}_{i2}|\omega) \le 1$, where $\omega = \{1, 1, \cdots, 1\}$. Then $\boldsymbol{VG}_{(i-1)p} = \{vg_{(i-1)p0}, vg_{(i-1)p1}, \cdots, vg_{(i-1)pn}\}$ of $\boldsymbol{G}_{(i-1)p}$ can be calculated by: if $vg_{i1} \oplus vg_{i2} = 0$, $vg_{(i-1)pj}$=vg_{i1j}, else $vg_{(i-1)pj}$="*".

In granular space, each granule has only one feature vector $\boldsymbol{VG}_{ij}$, so $\boldsymbol{VG}_{ij}$ can be viewed as the key of $\boldsymbol{G}_{ij}$. This viewpoint is ensured by Proposition 3.2.

3.2 Decision Table and Decision Rule

In above description of problem, we mention that $\boldsymbol{OS}_1$ is a decision table and the adjoint vector of each granule can be viewed as a decision rule. For the convenience of description, we define the two concepts clearly.

Definition 3.5. A decision table is defined as $\boldsymbol{S} =< \boldsymbol{U}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{V}, \boldsymbol{f} >$. The details can be found in Ref. [8].

Definition 3.6. Let $\boldsymbol{S} =< \boldsymbol{U}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{V}, \boldsymbol{f} >$ be a decision table, and let $\boldsymbol{B} \subseteq \boldsymbol{C}$. Then the rule set $\boldsymbol{F}$ generated from $\boldsymbol{S}$ and $\boldsymbol{B}$ consists of all rules of the form

$$\wedge\,(\boldsymbol{a}, \boldsymbol{v}) : \boldsymbol{a} \in \boldsymbol{B} \text{ } and \text{ } \boldsymbol{v} \in V_a \cup \{*\} \rightarrow \boldsymbol{d} = v_d, \tag{9}$$

where $v_d \in \boldsymbol{V}_a$. The symbol * means that the value of the corresponding attribute is irrelevant for the rule, i.e., in conjunction we are not considering the descriptor for this attribute $\boldsymbol{a}$. The length of the rule is the number of attribute values in precondition that not equal to "*", denoted as $||\cdot||$.

For example, in a decision system with 5 condition attributes $(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_5)$, $(\boldsymbol{a}_1 = 1) \wedge (\boldsymbol{a}_2 = *) \wedge (\boldsymbol{a}_3 = 1) \wedge (\boldsymbol{a}_4 = 1) \wedge (\boldsymbol{a}_5 = *) \rightarrow d = 4$ is a rule according to definition 3.6. In this paper, we describe a rule as a vector and the above decision rule is described as (1, *, 1, 1, *, 4), and the length of the rule is 3.

3.3 Decision Granule

Now, we choose some granules from the constructed granular space to solve our problem, called decision granules.

Definition 3.7. Let $\boldsymbol{S}$ be a decision table and $\boldsymbol{G}_{ij}$ describes a granule. $\boldsymbol{G}_{ij}$ is a decision granule, iff $\boldsymbol{G}_{ij}$ satisfies the following conditions:
(1) The objects in $\boldsymbol{G}_{ij}$ satisfy a tolerance proposition(Def 2.4-2.5) ;
(2) $\boldsymbol{VG}_{ij}$ (Def. 3.6) is a decision rule defined at definition 3.6;
(3) There isn't any object in $\boldsymbol{S}$ satisfying the condition of $\boldsymbol{VG}_{ij}$, but not satisfying the decision of $\boldsymbol{VG}_{ij}$.

Definition 3.8. Let $\boldsymbol{S}$ be an decision table, then $\boldsymbol{G}_*$ denotes the maximal decision granule, where $\boldsymbol{VG}_*$ is the rule covered all objects; $\boldsymbol{G}_{\Phi}$ denotes the minimal decision granule, where $\boldsymbol{VG}_{\Phi}$ is the rule covered none objects.

3.4 Decision Granular Lattice

Definition 3.9. Suppose $\boldsymbol{G}_{ij}$ and $\boldsymbol{G}_{kp}$ are two decision granules, where $\boldsymbol{k} \geq \boldsymbol{i}$. Then, we denote that $\boldsymbol{G}_{ij} \subseteq \boldsymbol{G}_{kp}$, iff $\boldsymbol{dis}(VG_{ij}, \boldsymbol{VG}_{kp}|\omega) = 0$ and $||\boldsymbol{VG}_{ij}|| \geq ||\boldsymbol{VG}_{kp}||$. Here, we call $\boldsymbol{G}_{ij}$ is the child granule of $\boldsymbol{G}_{kp}$, and $\boldsymbol{VG}_{kp}$ is the ancestral granule of $\boldsymbol{VG}_{ij}$. If $||\boldsymbol{VG}_{ij}|| = ||\boldsymbol{VG}_{kp}||+1$, we call $\boldsymbol{VG}_{ij}$ is the son granule of $\boldsymbol{VG}_{kp}$, and $\boldsymbol{VG}_{kp}$ is the father granule of $\boldsymbol{VG}_{ij}$.

Proposition 3.1. The relation "$\subseteq$"(**Def.3.6**) is a partial ordering relation.

Proposition 3.2. Let $\boldsymbol{S} =< \boldsymbol{U}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{V}, \boldsymbol{f} >$ be a decision table, $\boldsymbol{GS}$ is the set of all decision granules generated from $\boldsymbol{S}$ (including $\boldsymbol{G}_*$ and $\boldsymbol{G}_{\Phi}$) and "$\subseteq$" is the relation defined by definition 3.9, then $< \boldsymbol{GS}, \subseteq >$ is a lattice, called decision granular lattice.

For the sake of pages, we aren't proving these propositions here.

Definition 3.10. Suppose $\boldsymbol{G}_{ij}$ and $\boldsymbol{G}_{pg}$ are two decision granules, $\boldsymbol{G}_{ij}$ is conflict with $\boldsymbol{G}_{pg}$, iff

$$\sum_{k=0}^{1} \omega_j (vg_{ijk} \oplus vg_{pqk}) = 0 \tag{10}$$

and the decisions of $\boldsymbol{VG}_{ij}$ and $\boldsymbol{VG}_{pq}$ are different.

Definition 3.11. Suppose $\boldsymbol{G}_{ij}$ and $\boldsymbol{VG}_{pq}$ are two decision granules, $\boldsymbol{G}_{ij}$ is equal to $\boldsymbol{G}_{pg}$, iff $\boldsymbol{dis}(\boldsymbol{VG}_{ij},\ \boldsymbol{VG}_{pq}|\omega)$=0.

Now, we present a simple algorithm generating granular lattice from decision table $\boldsymbol{S}$. Suppose $\boldsymbol{m}$ is the number of condition attributes and $\boldsymbol{n}$ is the number of objects in $\boldsymbol{S}$, $\boldsymbol{object}_i$ is the $\boldsymbol{i}$th object in $\boldsymbol{S}$.

Algorithm 1 Decision Granular Lattice Construction Algorithm (DGLC)
Input: Decision table $\boldsymbol{S}$;
Output: Granular Lattice $\boldsymbol{GL}$;
Step 1(Initialization):
FOR($\boldsymbol{i} = 1; \boldsymbol{i} \leq \boldsymbol{n}; \boldsymbol{i}++$)
{Generate $\boldsymbol{G}_i$, where $\boldsymbol{VG}_i$=$\boldsymbol{object}_i$;
Add $\boldsymbol{G}_i$ to $\boldsymbol{GL}$; }
$Number\_of\_granules = \boldsymbol{n}$;
//In the following, we call $\boldsymbol{G}_i$ is the $\boldsymbol{i}$th granule in $\boldsymbol{GL}$ if the granule is the $\boldsymbol{i}$th granule added to $\boldsymbol{GL}$.
Step 2(Generating granules and establish relationships):
$\boldsymbol{Start}$=1; $\boldsymbol{End}$=$\boldsymbol{n}$;
While ($\boldsymbol{Start} \leq \boldsymbol{End}$)
{FOR ($\boldsymbol{i}$= $\boldsymbol{Start}$; $\boldsymbol{i} \leq \boldsymbol{End} - 1; \boldsymbol{i}++$){
FOR ($\boldsymbol{j}$=$\boldsymbol{i}$+1; $\boldsymbol{j} \leq \boldsymbol{End}; \boldsymbol{j}++$){
Suppose $\boldsymbol{G}_i$ is the $\boldsymbol{i}$th granule and $\boldsymbol{G}_i$ is the $\boldsymbol{j}$th;
Generate $\boldsymbol{G}$ and $\boldsymbol{VG}$ from $\boldsymbol{G}_i$ and $\boldsymbol{G}_j$ according to Def. 3.7;
If $\boldsymbol{G}$ exist and there isn't any conflict granule with $\boldsymbol{G}$, then {
If there isn't any granule equal to $\boldsymbol{G}$
{Add $\boldsymbol{G}$ to $\boldsymbol{GL}$; Connect $\boldsymbol{G}$ to $\boldsymbol{G}_i$ and $\boldsymbol{G}_j$;
$Number\_of\_granules$++;}
else Connect granule ***G'*** that is equal to $\boldsymbol{G}$ (Def. 3.11) to $\boldsymbol{G}_i$ and $\boldsymbol{G}_j$;}}}}
$\boldsymbol{Start}$=$\boldsymbol{n}$+1; $\boldsymbol{End}$=$Number\_of\_granules$;}
Step 3(Establishing the remaining relationships)
Connect the maximal decision granule to the granules without father;
Connect the granules without son to the minimal decision granule.
End

Example 1: Table 1 is a decision table and the last attribute is the decision attribute. Input table 1 to **DGLC** algorithm and **Fig.** 1 is the resulted decision granular lattice.

Table 1. A Decision Table

Object	F	R	E	S
X_1	f_1	r_1	e_1	s_1
X_2	f_2	r_1	e_3	s_1
X_3	f_2	r_1	e_2	s_0
X_4	f_3	r_2	e_1	s_1
X_5	f_1	r_1	e_2	s_1

3.5 Decision Granular Lattice Based Classification Algorithm

We can use decision granular lattice to classify new decision objects and the following is the decision granular lattice based classification algorithm (DGLC).

Definition 3.12. The matching degree of object $\boldsymbol{object}$ to a decision granule $\boldsymbol{G}$ is defined as Match($\boldsymbol{object}$, $\boldsymbol{G}$), and Match($\boldsymbol{object}$, $\boldsymbol{G}$)=max{$\boldsymbol{level}$| $\boldsymbol{level}$ is the level number of $\boldsymbol{G}$ or $\boldsymbol{G}$'s child granule $\boldsymbol{G}_{ij}$, where object satisfy $\boldsymbol{VG}$ or $\boldsymbol{VG}_{ij}$}.

Algorithm 2 Decision Granular Lattice Based Classification Algorithm (DGLBC)
Input: Testing table $\boldsymbol{S}$, decision granular lattice $\boldsymbol{GL}$;
Output: Classification Result
FOR ($\boldsymbol{i}$=1; $\boldsymbol{i} \leq \boldsymbol{n}$; $\boldsymbol{i}$++)//$\boldsymbol{n}$ is the object number of $\boldsymbol{S}$
{ $\boldsymbol{Max} = 0$; $\boldsymbol{MatchGranule} = \Phi$;
FOR (each son granule $\boldsymbol{G}$ of $\boldsymbol{G}_*$ (Def.3.11))
{ If $\boldsymbol{VG}$ cover $\boldsymbol{object}_i$,
$\boldsymbol{Y}$= Match ($\boldsymbol{object}_i$, $\boldsymbol{G}$);
If ($\boldsymbol{Y} > \boldsymbol{Max}$) $\boldsymbol{Max}$=$\boldsymbol{Y}$; $\boldsymbol{MatchGranule}$= $\boldsymbol{G}$;}}}
Decision of $\boldsymbol{object}_i$=the decision of the adjoint vector of $\boldsymbol{MatchGranule}$;}

From the experiment results, we conclude that decision granular lattice can classify data with higher correct rate than most of other algorithms. In specially, decision granular lattice has very high classification correct rate when the training set is small. It is because decision granular lattice is a granulation knowledge structure, which not only includes the knowledge of the final results but includes the granulated knowledge with different granularities.

4 Conclusion

We basically construct a more uniform granulation model, which is established on both consecutive space and discrete attribute space and based on tolerance relation.

(1) A tolerance relation based granular space $\boldsymbol{TG}$, which is described as ($\boldsymbol{OS}$, $\boldsymbol{TR}$, $\boldsymbol{NTC}$), are modeled and constructed.

(2) An illustration of how to use the tolerance relation based granular space to represent and solve problems is presented.

(3) A decision granular lattice is developed. The lattice is a granulation knowledge structure, which not only includes the knowledge of the final results but includes the granulated knowledge at different granularities.

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